

Categorification and representation theory

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About me

- born in Neustadt Weinstrasse, Germany;



- went to high-school at Kaethe-Kollwitz-Gymnasium and Leibniz-Gymnasium there;



About me

- studied in Stuttgart and also did my PhD there;



- with a year abroad in Oregon, USA;



About me

- spent four years doing postdoctoral research in Oxford and Cologne;



- then moved to UEA Norwich, where I lecture and do research at the intersection of representation theory and category theory.



category



About me

- In my free time I like to swim, dance, climb and sing.



Categorification

What is categorification ?

Decategorification: forgetting information

Examples

set S	$\rightsquigarrow$	$ S $, a nonnegative integer
vector space V	$\rightsquigarrow$	$\dim V$, a nonnegative integer

Categorification: the opposite - not constructive!

Advantage: more information

Categories

What have we done?

Definition. A **category** $\mathcal{C}$ consists of

- a collection $\text{Ob } \mathcal{C}$ of objects
- for all $X, Y \in \text{Ob } \mathcal{C}$ a set of morphisms $\text{Hom}(X, Y)$
- for all $X, Y, Z \in \text{Ob } \mathcal{C}$ a binary operation (composition)

$$- \circ -: \text{Hom}(Y, Z) \times \text{Hom}(X, Y) \rightarrow \text{Hom}(X, Z),$$

which is associative

- for each $X \in \text{Ob } \mathcal{C}$ an identity morphism $1_X \in \text{Hom}(X, X)$ such that for all $Y \in \text{Ob } \mathcal{C}$ and all $f \in \text{Hom}(X, Y), g \in \text{Hom}(Y, X)$, we have $f \circ 1_X = f, 1_X \circ g = g$.

Categories

Examples.

- Set^f :
 - $Ob\,Set^f = \{S \mid S \text{ finite set}\},$
 - morphisms are all functions
- $Vec_{\mathbb{k}}^f$:
 - $Ob\,Vec_{\mathbb{k}}^f = \{V \mid V \text{ finite-dimensional } \mathbb{k}\text{-vector space}\},$
 - morphisms are all linear maps
- Grp^f :
 - $Ob\,Grp^f = \{G \mid G \text{ finite group}\},$
 - morphisms are group homomorphisms.
- For a ring R , we have $R\text{-mod}$:
 - $Ob\,R\text{-mod} = \{M \mid M \text{ finitely generated } R\text{-modules}\},$
 - morphisms are R -module homomorphisms.

Decategorification

Let $\mathcal{C}$ be a category. We denote by $K_0(\mathcal{C})$ the set of isomorphism classes of objects in $\mathcal{C}$.

Examples.

- $K_0(\mathcal{S}et^f) \cong \mathbb{N}_0$ via $[A] \mapsto |A|$
- $K_0(\mathcal{V}ec_{\mathbb{k}}^f) \cong \mathbb{N}_0$ via $[V] \mapsto \dim V$
- $K_0(\mathbb{Z}\text{-mod}) =$

$$\left\{ \mathbb{Z}^{\oplus r} \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_s\mathbb{Z} \mid \begin{array}{l} r, s \in \mathbb{Z}_{\geq 0} \\ n_i \in \mathbb{Z}_{>0} \\ n_i | n_{i+1} \forall i = 1, \dots, s \end{array} \right\}$$

Decategorification

Some categories have additional structure, e.g. a product/coproduct/biproduct. This induces operations on K_0 .

Examples.

- Set^f :
 - $|A \sqcup B| = |A| + |B|$,
 - $|A \times B| = |A||B|$
- $Vec_{\mathbb{k}}^f$:
 - $\dim(V \oplus W) = \dim V + \dim W$,
 - $\dim(V \otimes W) = \dim V \cdot \dim W$

Categorification

An category $\mathcal{C}$ is **additive**, if

- morphism sets are abelian groups and composition is bilinear
- all finite (co)products exist in $\mathcal{C}$

Examples.

- $\mathcal{A}b = \mathbb{Z}\text{-Mod}$ the category of abelian groups
- $\mathcal{V}ec_{\mathbb{k}}^f$
- $R\text{-mod}$

If $\mathcal{C}$ is an additive category, $K_0(\mathcal{C})$ is a monoid. We add formal inverses to make it into an abelian group.

Categorification

How to categorify things?

Naive categorification:

- Suppose X is a set. Construct a category $\mathcal{X}$ with $K_0(\mathcal{X}) = X$ as follows:
 - $Ob \mathcal{X} = X$
 - for $x, y \in X$, $Hom(x, y) = \begin{cases} \emptyset & \text{if } x \neq y \\ \{1_x\} & \text{if } x = y \end{cases}$

Then $K_0(\mathcal{X}) = X$.

- Suppose A is a free abelian group on generating set X . Construct $\mathcal{A}$ as the additive closure of $\mathcal{X}$:
 - $Ob \mathcal{A} = \{(x_1, \dots, x_n) \mid x_i \in X, n \in \mathbb{Z}_{\geq 0}\}$
 - morphisms are matrices (f_{ij}) for $f_{ji} \in Hom(x_j, x_i)$, composition given by matrix multiplication

Then $K_0(\mathcal{A}) = A$.

This is obviously boring - no extra information!

Representation theory

More interesting examples? Some representation theory...

Let R be a ring. A left R -module is an abelian group $(M, +)$ together with an action

$$R \times M \rightarrow M, (a, m) \mapsto am$$

such that $\forall r, s \in R, m, n \in M$

- $(r + s)m = rm + sm$
- $r(m + n) = rm + rn$
- $(rs)m = r(sm)$
- $1_R m = m.$

Right modules are defined similarly with an action $M \times R \rightarrow M$. An R - R -bimodule is a left and right R -module such that

$$\forall r, s \in R, m \in M \qquad r(ms) = (rm)s.$$

Representation theory

Examples.

- $\mathbb{Z}$ -module = abelian group
- $\mathbb{k}$ -module = $\mathbb{k}$ -vector space
- R is an R - R -bimodule by left and right multiplication

$R\text{-mod}$, $\text{mod-}R$, $R\text{-mod-}R$:

categories of left/right R -/ R - R -bimodules with module homomorphisms (obvious definition - preserves the structure: $\phi(rm) = r\phi(m)$)

Given two R - R -bimodules M and N , define their tensor product

$$M \otimes_R N = \{(m, n) \mid m \in M, n \in N\} / \{(mr, n) - (m, rn) \mid r \in R\}$$

Example. $R \otimes_R M \cong M$ via $1 \otimes m \leftrightarrow m$

Soergel bimodules

Let $R = \mathbb{C}[x_1, \dots, x_n]$.

The symmetric group S_n acts on R by permuting variables.

For a simple transposition $s \in S_n$, let

$$R^s = \{f \mid s(f) = f\}.$$

This is a subring of R .

Set $B_s = R \otimes_{R^s} R$. The category $\mathcal{S}$ of Soergel bimodules is generated by $\{B_s \mid s \text{ simple transposition in } S_n\}$ under $- \otimes_R -$, direct sums and direct summands (and grading shifts, ignored here).

Theorem. [Soergel]

$K_0(\mathcal{S})$ is isomorphic to the Hecke algebra for S_n .

Soergel bimodules

Remarks.

- This works more generally for any Coxeter group. [*Soergel, Elias–Williamson*]
- Hecke algebras show up in many places: representation theory of finite or algebraic groups, mathematical physics, knot theory, topological quantum computation, etc.
- There are other interesting categorifications, e.g. of Schur algebras, quantum groups, Lie superalgebras, Hall algebras, etc.

What is this useful for?

- positivity of coefficients in Kazhdan–Lusztig polynomials
- new knot invariants
- counterexamples to Lusztig's conjectures for characters of simple modules for algebraic groups in positive characteristic

Open questions.

- Describe the indecomposable objects in $\mathcal{S}$ explicitly.
- There is a braid group action on the bounded homotopy category of Soergel bimodules. Is it faithful even in affine types?
- Understand Soergel bimodules over a field of positive characteristic.
- Understand the (2) -representation theory of Soergel bimodules.

End

Thank you for your attention!