Dispersion and discontinuities in mathematics and in life

Beatrice Pelloni
Heriot Watt University & Maxwell Institute - Edinburgh

PiWORKS Hybrid Seminar, May 2026

in collaboration with mentors, colleagues, and students

About me - CV in a nutshell

PhD - Yale 1996

I had a child (eventually 3 children) not long after starting, so took a long time to finish. *Thank you to Yale and the US system* that allowed me to take the time I needed.
Thank you to *V. Dougalis* - crucial mentor

Marie Curie Fellowship - 1997

This was the turning point for my career. It took me to Imperial College. I stayed on for a second PDRA position until 2001

Reading, 2001-2016

Lecturer, then Reader (2005), then Professor (2012) and HoD at the University of Reading. I was also Director of the CDT MPE

Maxwell Institute, since 2016

Professor at Heriot-Watt
I was Head of School 2016-2022. I am currently Deputy Director at ICMS and President of the Edinburgh Mathematical Society

A quick summary of my research work

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Models of fluid dynamics - particularly dispersive waves, linear and nonlinear, and lately geophysical flows

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Methods in spectral and lately also variational analysis

- *Unified Transform*, a generalised Fourier transform for solving linear and integrable nonlinear boundary value problems - met at the start of my Marie Curie fellowship (*T.Fokas*)
- *Optimal transform methods*, to study problems defined by an energy minimisation principle - met as I started my HoD tenure (*M. Cullen*)

Both areas I had worked in before becoming HoS, when I hit the second big discontinuity

Smoothing discontinuities

At critical times, I was supported by understanding colleagues and was lucky to find new problems.

After finishing as HoS, I had a chance encounter, in a seminar talk by *P. Olver*, with a new problem, with vague connections with problems I had worked or studied before:

the Talbot effect

In this talk I will describe mostly the work on this effect, a more recent interest that revitalised my research activity after I finished my term as HoS

Some of my results

1. **2-point boundary value problems**, such as

$$u_t + u_{xxx} = 0, \qquad t > 0,\ 0 < x < L$$

with 3 boundary + 1 initial conditions.

For $u(0,t) = u(L,t) = 0$, $u_x(0,t) = 0 \to$ cannot be solved by Fourier series like the heat equation! The explicit solution via the Unified Transform involves complex contour integrals

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2. **Large-scale atmospheric flow models** - analysis via optimal transport to prove existence of solution that lack regularity, numerics to model cyclone formation via semi-discrete optimal transport.
3. **Dispersive revivals**, aka Talbot effect, discovering new revival phenomena.

A 2-point boundary value problem

Stokes/Airy equation

$$u_t = u_{xxx}, \quad u(x,0) = u_0(x), \quad u(0,t) = u(1,t) = u_x(0,t) = 0.$$

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Solution: complex contour integral

$$\begin{aligned} 2\pi u(x,t) &= \int_{\mathbb{R}} \mathrm{e}^{-i\lambda^3 t + i\lambda x} \hat{u}_0(\lambda) d\lambda + \int_{\partial D^+} \mathrm{e}^{-i\lambda^3 t + i\lambda x} \frac{\tilde{F}(\lambda)}{\Delta(\lambda)} d\lambda \\ &\quad + \int_{\partial D^-} \mathrm{e}^{-i\lambda^3 t + i\lambda(x-1)} \frac{\tilde{G}(\lambda)}{\Delta(\lambda)} d\lambda \end{aligned}$$

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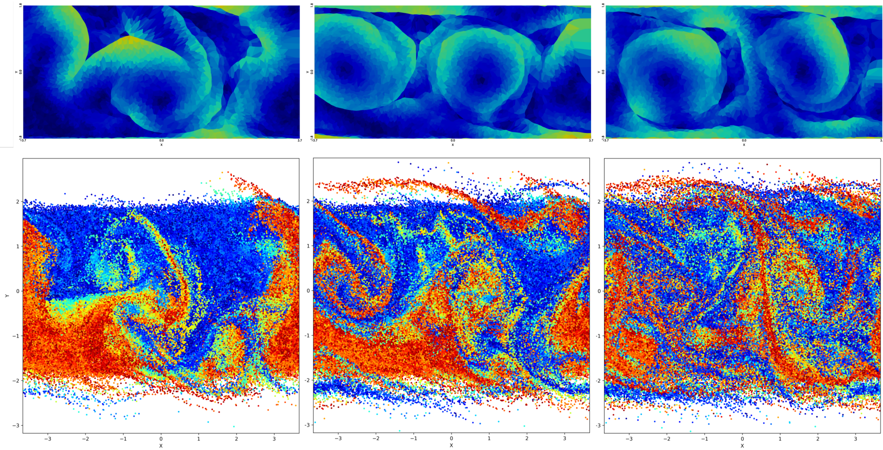
Heat equation:

$$u_t = u_{xx}, \quad u(x,0) = u_0(x), \quad u(0,t) = u(1,t) = 0$$

$$\rightarrow \quad u(x,t) = \sum_{n=1}^{\infty} \hat{u}_0^{sin}(n) \sin(\pi n x) e^{-n^2 \pi^2 t}$$

A semigeostrophic cyclone - optimal transport solution

$t \approx 16$ Days $t \approx 20$ Days $t \approx 25$ Days

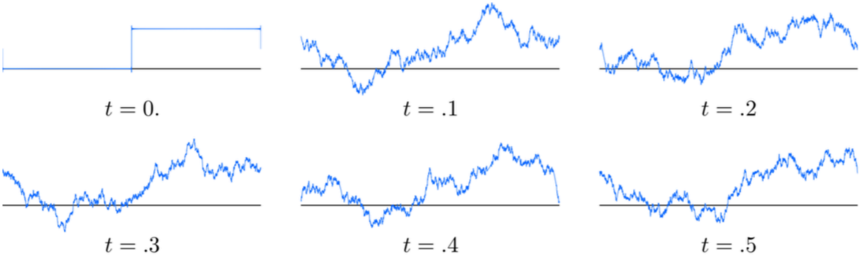


Velocity and temperature of a twin cyclone

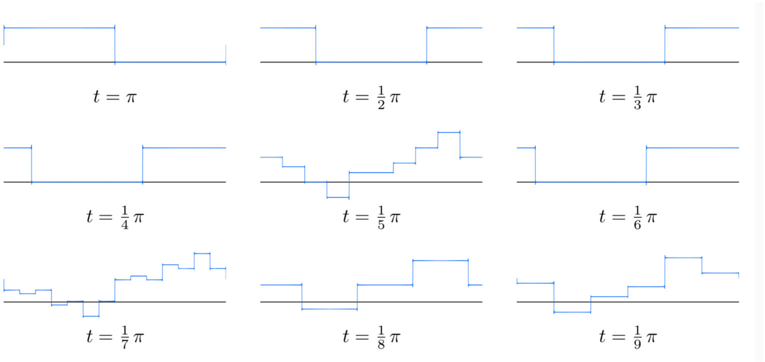
Using a semi-discrete optimal transport scheme, 36k points

Dispersive revivals

In a seminar talk, Peter Olver showed these pictures for the solution at various time of the 2π-periodic Airy equation $u_t = u_{xxx}$ starting from a step function:



...and then the solution at special values of the time



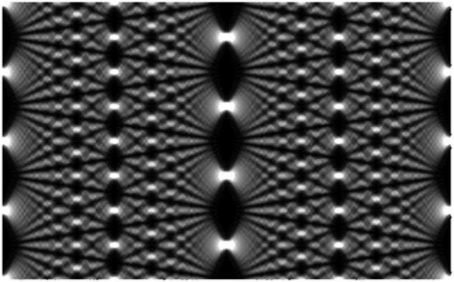
Rational times

A step back: Talbot and his optical effect

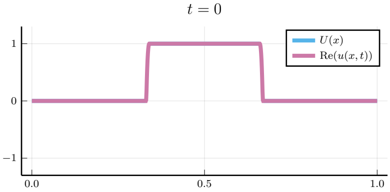
Talbot observations in 1835:

Observing light passing through a diffraction grating, he observed that, at each **rational** *multiple of a fixed distance, the diffraction pattern appears to reproduce a finite number of copies of the grating pattern.*

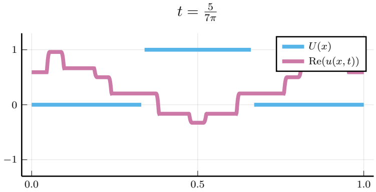
Talbot effect: the *self-imaging* of a diffraction grating. At regular distances from the grating, the light diffracted through it forms a nearly perfect image of the grating itself.



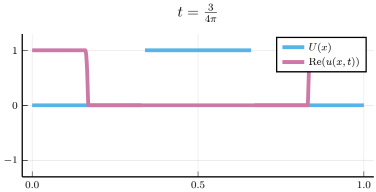
Free-space Schrödinger on $\mathbb{T}$ - step initial condition on $[0, 1]$



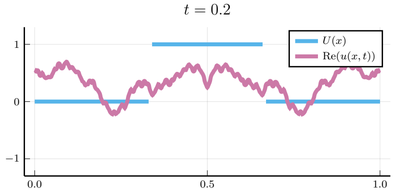
(a) Step initial condition



(b) Solution at $t = 5/7\pi$

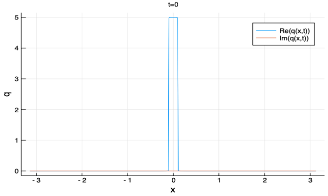


(c) Solution at $t = 3/4\pi$

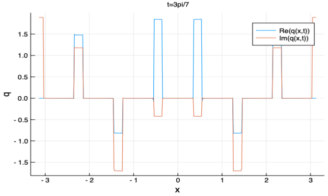


(d) Solution at irrational time $t = 0.2$

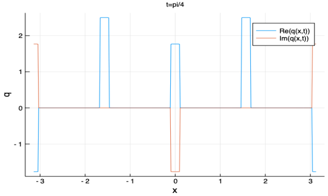
Free-space Schrödinger on $\mathbb{T}$ - box initial profile



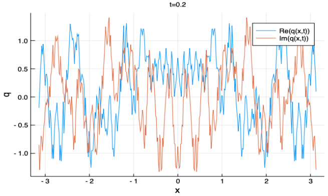
(a) Narrow box initial condition



(b) Solution at $t = 3\pi/7$



(c) Solution at $t = \pi/4$



(d) Solution at $t = 0.2$

Periodic revivals/fractalisation - Schrödinger equation

$$\partial_t u(x,t) = i\partial_x^2 u(x,t) \qquad x \in \mathbb{T}, \quad t > 0$$
$$u(x,0) = u_0(x) \qquad x \in \mathbb{T}.$$

Theorem

Let $u_0 \in \mathrm{BV}(\mathbb{T})$. Then:

(a) At $t = 2\pi p/q$

$$u\Big(x, 2\pi\frac{p}{q}\Big) = \frac{1}{q}\sum_{k=0}^{q-1}\left[\sum_{m=0}^{q-1} \mathrm{e}^{2\pi i \frac{km}{q}}\,\mathrm{e}^{2\pi i \frac{p}{q} m^2}\right] u_0\Big(x - 2\pi\frac{k}{q}\Big)$$

for co-prime $p, q \in \mathbb{N}$ (u_0 is *revived* if $t \in 2\pi\mathbb{Q}$);

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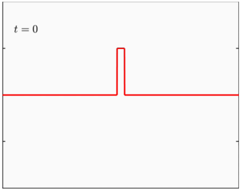
(b) **but...** $Re(u)$, $Im(u)$ are continuous in x for $t \notin 2\pi\mathbb{Q}$ (continuous also in t if u_0 is continuous);

(c) if $u_0 \notin H^s(\mathbb{T})$, $s > \frac{1}{2}$, for almost all $t > 0$ the graph of both $Re(u)$ and $Im(u)$ has fractal dim $= \frac{3}{2}$ (*fractalisation*).

True also for the nonlinear PDE- NLS: $iu_t + u_{xx} + |u|^2 u = 0$

Theorem by Erdogan-Tzirakis stating a *weak revival property*



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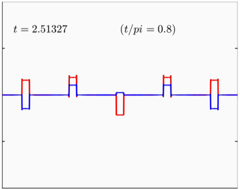
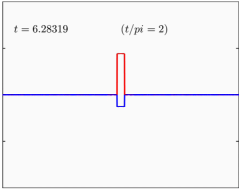
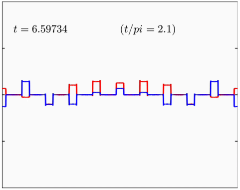


(b)



(c)



(d)

In summary: Periodic revivals

Periodic revivals: the solution of a linear dispersive periodic problem, at times equal to rational multiples of (*a constant depending on*) the period, is a finite linear combination of translated and reflected copies of the initial profile

[1] M. B. Erdoğan, N. Tzirakis, *Dispersive PDEs*, (Cambridge University Press, 2016)

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Theorem[1] Consider the dispersive PDE

$$\partial_t u(x,t) = iP(-i\partial_x)u(x,t), \quad u(x,0) = u_0(x), \quad x \in \mathbb{T}$$

$P(k)$ a polynomial with integer coefficients.

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At $t = 2\pi \frac{p}{q}$, the solution admits the representation

$$u(x, 2\pi\frac{p}{q}) = \frac{1}{q}\sum_{k=0}^{q-1} G_{p,q}(k) u_0(x - 2\pi\frac{k}{q}),$$

$$G_{p,q}(k) = \sum_{m=0}^{q-1} e^{-2\pi i P(m)\frac{p}{q}} e^{2\pi i m \frac{k}{q}}.$$

The proof is based on elementary number-theoretic properties.

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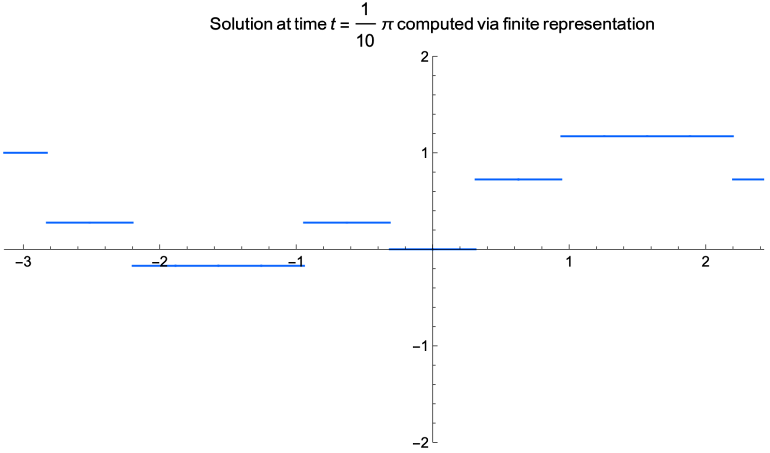
Finite sum solution of Airy - step initial condition



In summary: Fractalisation

Fractalisation: At *irrational* times (hence a.e. in $t > 0$) the solution, starting from a BV (*hence possibly discontinuous*) initial profile, is a continuous function of x whose graph has fractal dimension > 1 - ($\frac{3}{2}$ for Schrödinger, in $[\frac{5}{4}, \frac{7}{4}]$ for Airy).

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[2] H. L. Montgomery, *Ten lectures on the interface between analytic number theory and harmonic analysis*, (American Mathematical Soc., 1994)

[3] V. Chousionis *et al.*, *Proceedings of the London Mathematical Society* **110**, 543–564 (2014)

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Hence **the solution has better regularity properties at irrational than at rational times**.

The *proof* is based on number theoretical results[2] and on regularity estimates in Besov spaces[3].

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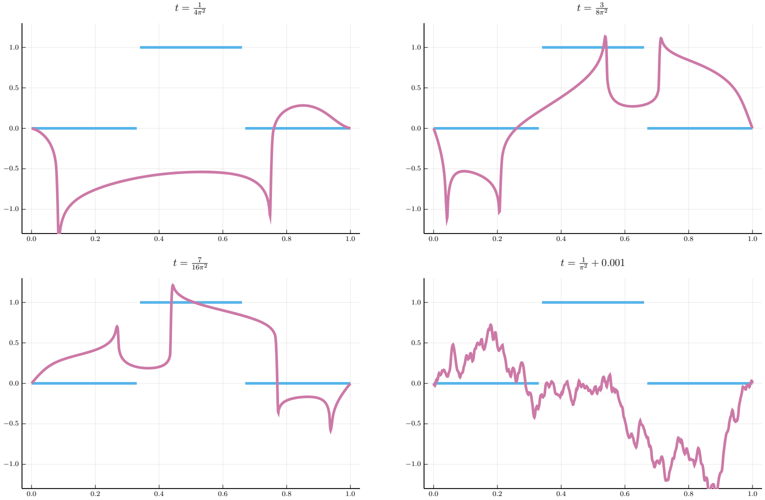
What about the periodicity? Airy with $u(0,t) = u(1,t) = 0$, $u_x(0,t) = u_x(1,t)$



blue: *initial condition* - magenta: *exact solution*

What is going on?

The answer is hidden in the spectral asymptotics of the spatial operator and the interaction with the periodic Hilbert transform[4]

[4] *L. Boulton, G. Farmakis, BP and D.A. Smith, ArXiv preprint:* 2403.01117

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Periodic Hilbert transform $\mathcal{H}$ on $[0,1]$:

for

$$f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{2\pi i n x}, \qquad f \in L^2[0,1]$$

then

$$\mathcal{H}f(x) = i \sum_{n=1}^{\infty} \Big[\widehat{f}(-n) e^{-2\pi i n x} - \widehat{f}(n) e^{2\pi i n x} \Big].$$

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Crucial if elementary identity:

$$\sum_{n=1}^{\infty} \widehat{f}(n) e^{2\pi i n x} = \frac{(Id + i\mathcal{H})f - \hat{f}(0)}{2}$$

[4] *L. Boulton, G. Farmakis, BP and D.A. Smith, ArXiv preprint:* 2403.01117

Periodic Hilbert transform $\mathcal{H}$ of a step function

$\mathcal{H}$ the periodic Hilbert transform:

$$\mathcal{H}g(x) = \frac{1}{2\pi}\ \text{p.v.}\int_{-\pi}^{\pi} \cot\frac{x-y}{2} g(y)\,\mathrm{d}y \Longrightarrow \widehat{\mathcal{H}g}(k) = -i\,sgn(k)\hat{g}(k).$$

$$\widehat{iu_{xx}} = -ik^2\hat{u}(k) \quad vs \quad \widehat{\mathcal{H}u_x x}(k) = -ik|k|\hat{u}(k)$$

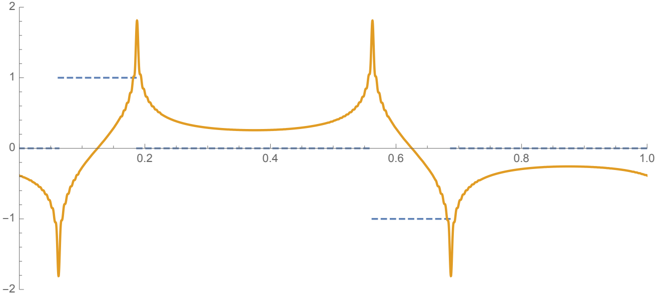
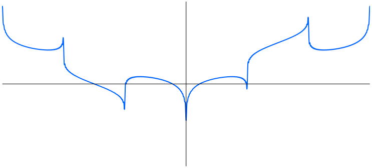
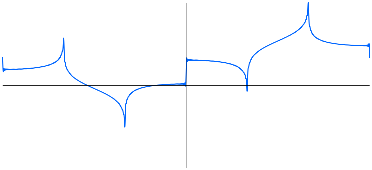


Figure: A step function (dashed) and its periodic Hilbert transform (solid). Where the given profile has a point of discontinuity, its periodic Hilbert transform displays an (infinite) logarithmic cusp

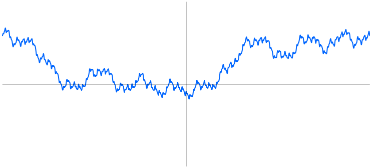
The linearised Benjamin-Ono equation $u_t = \mathcal{H}u_{xx}$ periodic, step initial condition



$t = \frac{\pi}{3}$



$t = \frac{\pi}{6}$



$t = 0.9$

The periodic linearised BO equation

$$BO: \ \widehat{iu_{xx}} = -ik^2\hat{u}(k) \quad vs \quad lS: \ \widehat{\mathcal{H}u_{xx}}(k) = -ik|k|\hat{u}(k)$$

Hilbert transform identity: for $g \in L^2(\mathbb{T})$

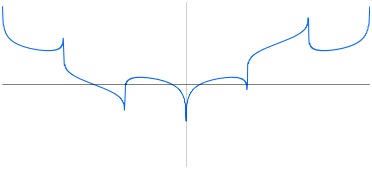
$$\sum_{n=1}^{\infty} \hat{g}(n)e_n(x) = \frac{(Id + i\mathcal{H})g - \langle g \rangle}{2},$$

Lemma *Assume* u_0 real-valued, *and WLOG* $\langle u_0 \rangle = 0$.
For u *solution of linear BO,* v *solution of free-space Schrödinger*

$$u(x,t) = Re\left[(Id + i\mathcal{H})v(x,t)\right].$$

This implies the result on (cusp) revivals, both for the continuous/discontinuous dychotomy and the fractal dimension
The weak version appears to hold for the full nonlinear problem

Linearised BO, 2π-periodic, step initial condition



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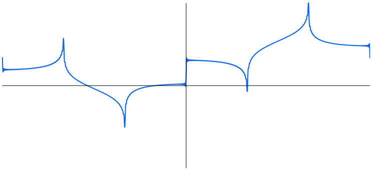

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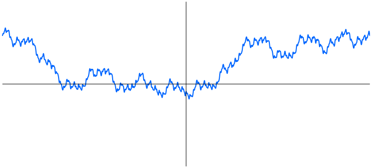


$t = 0.9$

Numerical evaluation - fractal (box-counting) dimension

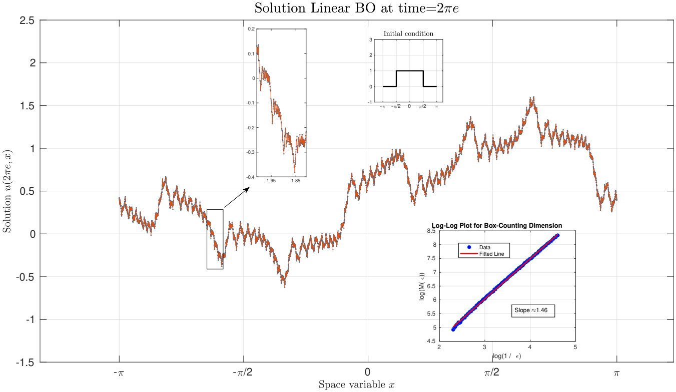
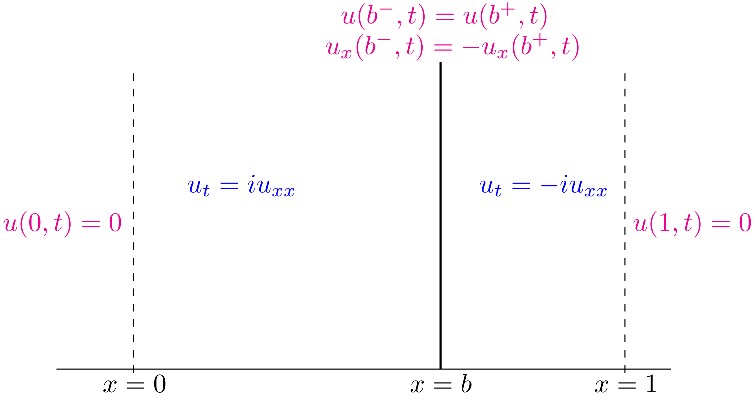


Figure: A step function (dashed) and its periodic Hilbert transform (solid). When a given profile has a point of discontinuity, its periodic Hilbert transform displays a logarithmic cusp [5]

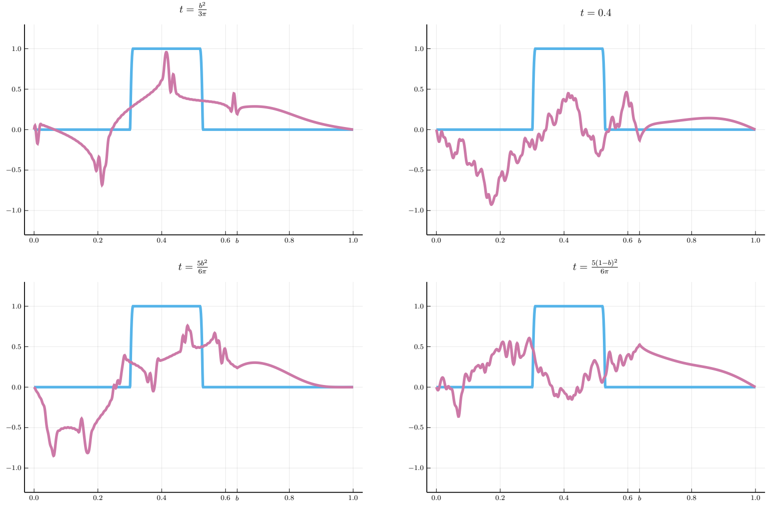
[5] *L. Boulton, B. Macpherson, BP, ArXiv preprint:* 2501.01322

More surprises: Schrödinger with a dislocation at $x = b$



Dislocation model - step initial condition, $b = 0.636619$

rational /irrational times, initial discontinuities to the left of b, 250 modes



blue: *initial condition* - magenta: *exact solution*

Summary: dispersive revivals

- for linear dispersive PDEs, **periodic**, initial discontinuities are propagated in the solution for a (measure zero) set of special values of the time
  but for almost all times the solution is continuous
- *polynomial dispersion*: jumps stay jumps
  non-polynomial dispersion of $d^0 \geq 2$: jumps may become cusps
- robust phenomenon that survives (in a weaker form) the perturbation by nonlinearity, quasi-periodicity, stochastic noise
- it can also survive, in weak cusp form, when the boundary conditions are not periodic
- Applications!?!

Thank yous and final remarks ....

- Conventional ways are not always the best to achieve a goal, or not the best way for you - do not hesitate to do things your own way
- Follow your instinct and principles, if you feel strongly about them.
- This is a time to fight for fairness, lateral thinking and plurality of views - make your voice heard!

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