

Hidden Patterns in Numbers

Himanshi Chanana

PiWORKS Seminar

Queen Mary University of London

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Are Prime Numbers Random?

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Proof. Suppose **NOT** and p_1, p_2, \dots, p_n . Consider the number $N = p_1 p_2 \cdots p_n + 1$. Observe $p_i \nmid N$ for all i . Therefore, N is not divisible by any of the primes in our list.

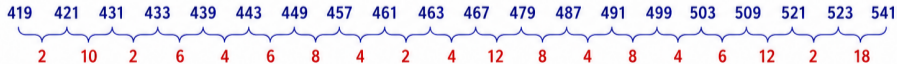
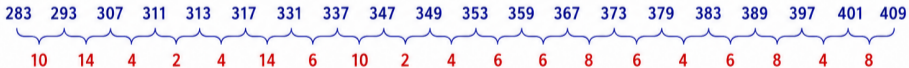
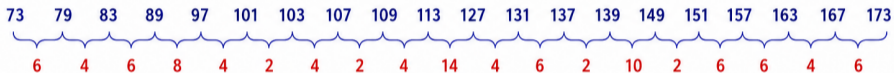
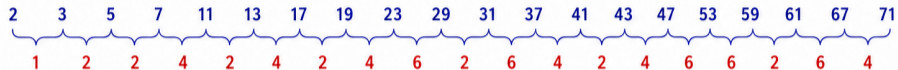
This implies that either:

- N is itself prime, or
- N has a prime divisor not in the list.



Gaps Between Consecutive Primes

Gap = difference between consecutive primes



- Smallest gap: **1** (between 2 and 3)
- Largest gap (shown): **18** (between 523 and 541)
- Average gap (shown): **4.84**
- Gaps are always even, except for the first gap (1).

Can we detect any pattern in the distribution of primes?

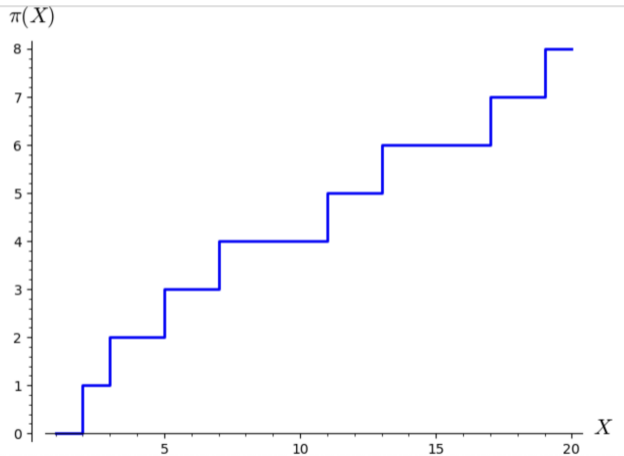
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$$\pi(x) := \sum_{\substack{p \leq x \\ p \text{ prime}}} 1$$

Prime Counting Function

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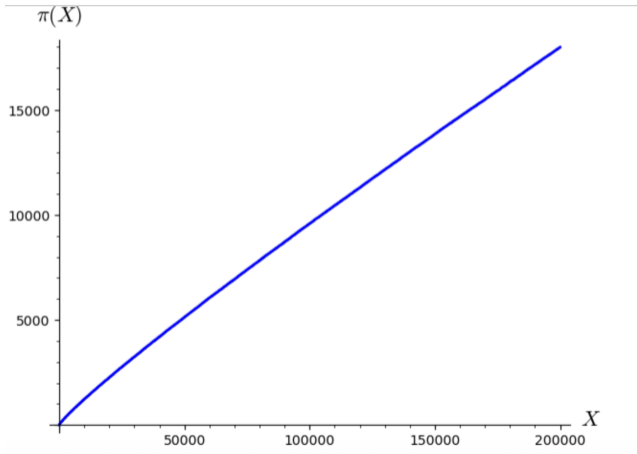


Figure: Carl Friedrich Gauss

Prime Number Theorem (Hadamard and de la Vallée Poussin, 1896)

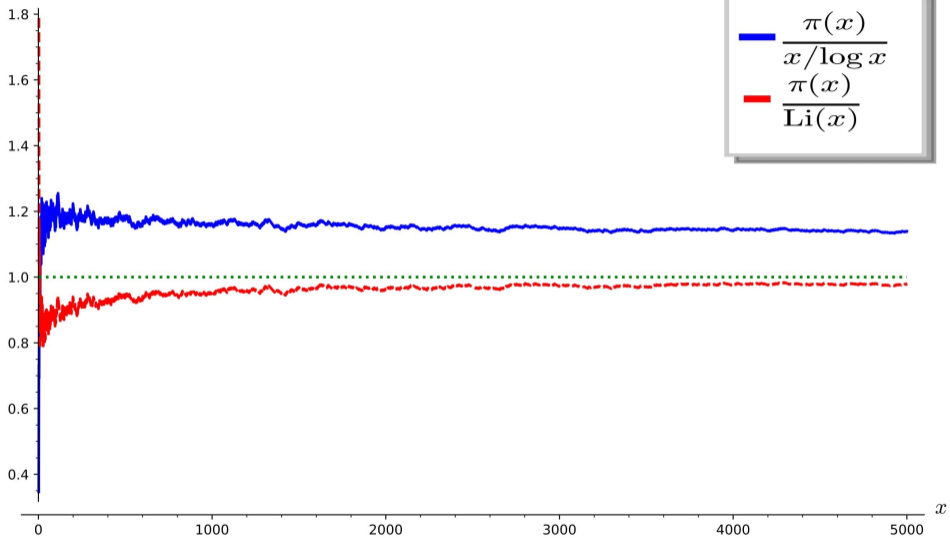
$$\pi(x) \sim \text{Li}(x) := \int_2^x \frac{dt}{\log t} \quad \text{as } x \rightarrow \infty.$$

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as } x \rightarrow \infty.$$

Equivalently,

$$\lim_{x \rightarrow \infty} \frac{\pi(x)}{x/\log x} = 1 \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{\pi(x)}{\text{Li}(x)} = 1$$

ratio



$\frac{\pi(x)}{x/\log x}$
 $\frac{\pi(x)}{\text{Li}(x)}$

Euler (18th century): Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots, \quad s > 1$$



Figure: Leonhard Euler

Euler (18th century):

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$



Figure: Leonhard Euler

Theorem (Euler Product)

For $s > 1$, we have

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$$\prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} = \prod_{p \text{ prime}} \sum_{k=0}^{\infty} \frac{1}{p^{ks}}.$$

Expanding this product formally, each term is of the form $\frac{1}{p_1^{k_1 s} p_2^{k_2 s} \cdots p_r^{k_r s}}$ where only finitely many k_i are nonzero. By the fundamental theorem of arithmetic, every integer $n \geq 1$ can be uniquely written as $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$.

□

Riemann (1859):

For $\Re(s) > 1$,

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$



Figure: Bernhard Riemann

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We study:

- where the function converges
- how it behaves
- where it becomes zero



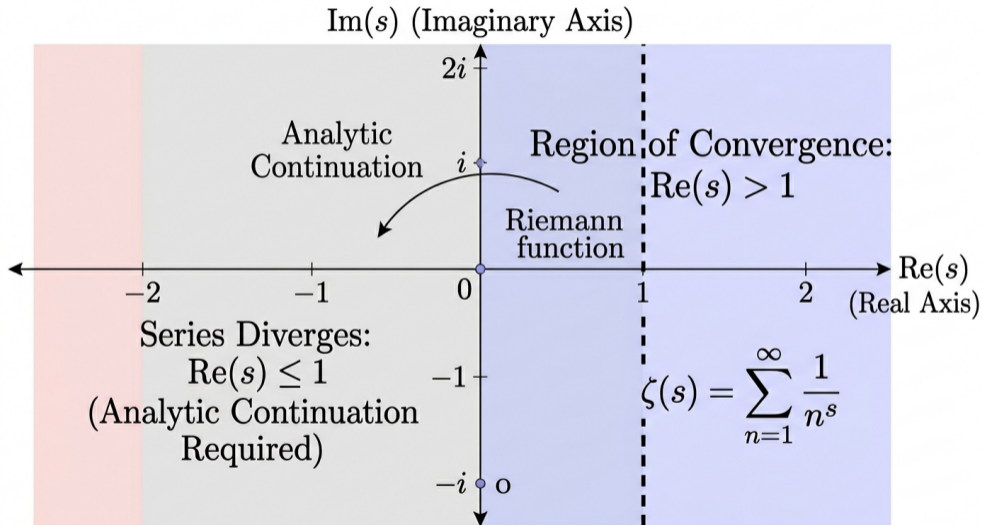
Figure: Bernhard Riemann

As $|n^{-s}| = n^{-\Re(s)}$. Therefore, $\zeta(s)$ converges absolutely for $\Re(s) > 1$.

Q: What will be $\zeta(s)$ for $\Re(s) \leq 1$?

.

- Analytic Continuation



- Analytic Continuation

Example: Take

$$1 + z + z^2 + z^3 + \dots \quad \text{valid for } |z| < 1.$$

Also,

$$1 + z + z^2 + \dots = \frac{1}{1 - z}.$$

But $\frac{1}{1-z}$ makes sense for $z \in \mathbb{C} \setminus \{1\}$.

- **Analytic Continuation**
- **Functional Equation**

$$\Gamma\left(\frac{s}{2}\right) \pi^{-s/2} \zeta(s) = \Gamma\left(\frac{1-s}{2}\right) \pi^{-(1-s)/2} \zeta(1-s) \quad \text{for all } s \in \mathbb{C}$$

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where $\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} dt$ is the **Euler Gamma function** defined for $\Re(s) > 0$.

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where $\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} dt$ is the **Euler Gamma function** defined for $\Re(s) > 0$ and $\Gamma(n) = (n-1)!$ for $n \in \mathbb{Z}$.

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Q: What about $\zeta(s)$ for $0 < \Re(s) \leq 1$?

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For $\Re(s) > 0$, $s \neq 1$

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{\infty} \frac{\{x\}}{x^{s+1}} dx$$

where $\{x\}$ is the fractional part of x (i.e., $x - \lfloor x \rfloor$).

- $\zeta(s)$ has a **simple pole** (singularity) at $s = 1$.

For $\Re(s) > 1$: Suppose $\zeta(s_0) = 0$, for some s_0

$$\frac{1}{\zeta(s_0)} = \prod_{p \text{ prime}} (1 - p^{-s_0})$$

FACT: $\prod(1 - a_n)$ converges to a **non-zero** value if the sum $\sum |a_n|$ converges.

Since $\sum |p^{-s_0}|$ converges for $\Re(s_0) > 1$, the product $\prod(1 - p^{-s_0})$ must converge to a non-zero limit. Therefore, $1/\zeta(s_0)$ cannot be infinite, and $\zeta(s_0)$ cannot be zero. **Contradiction.**

For $\Re(s) < 0$: Recall the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$$

1. $\Gamma(1-s)$: The Gamma function has no zeros. (It has poles at $0, -1, -2, \dots$, but since $\Re(s) < 0$, $\Re(1-s) > 1$, where Γ is well-behaved).
2. $\zeta(1-s)$: We have already proved that $\zeta(s) \neq 0$ for $\Re(s) > 1$.

Therefore, $\zeta(s) = 0$ if and only if $\sin\left(\frac{\pi s}{2}\right) = 0$.

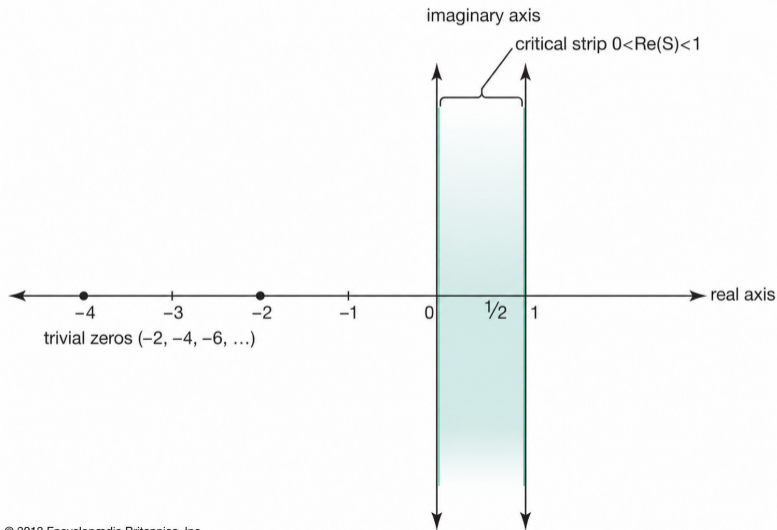
$$\frac{\pi s}{2} = k\pi \implies s = 2k, \quad k \in \mathbb{Z}$$

In the region $\Re(s) < 0$, the valid integers are $k = -1, -2, -3, \dots$

$s = -2, -4, -6, \dots$. These are the **trivial zeros**.

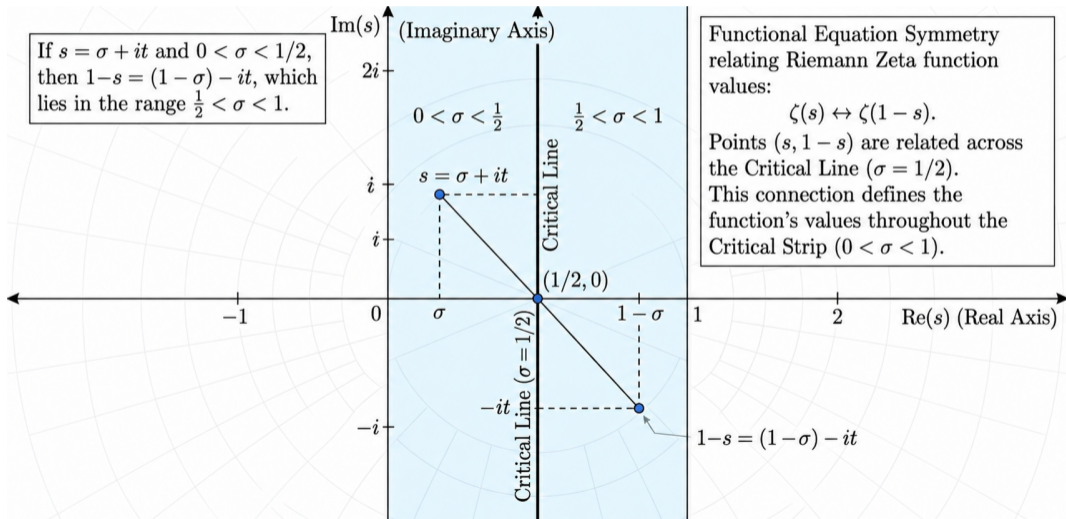
For $\Re(s) = 1$: $\zeta(1 + it) \neq 0$ for all real $t \neq 0$ (trust me on this!).

For $\Re(s) = 0$: FE $\implies \zeta(s) \neq 0$. Since if $s = it$ then $1 - s = 1 - it$.



The Riemann Hypothesis: All non-trivial zeros of $\zeta(s)$ lie on $\Re(s) = \frac{1}{2}$

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Q: How is $\zeta(s)$ related to distribution of primes?

Define (Chebyshev function)

$$\psi(x) := \sum_{n \leq x} \Lambda(n) = \sum_{p^m \leq x} \log p$$

where $\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m, m \geq 1 \\ 0 & \text{otherwise} \end{cases}$ is **Von Mangoldt function**.

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$$\pi(x) \sim \frac{x}{\log x} \iff \psi(x) \sim x$$

Von Mangoldt (1895):

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \log(1 - x^{-2}) \quad (x > 1, x \neq p^m),$$

where the sum is over **non-trivial zeros** $\rho = \beta + i\gamma$ of $\zeta(s)$.

Here $x^{\rho} = x^{\beta} e^{i\gamma \log x}$.

$$\zeta(1 + it) \neq 0 \text{ for } t \neq 0 \iff \pi(x) \sim \frac{x}{\log x}$$

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where the sum is over **non-trivial zeros** $\rho = \beta + i\gamma$ of $\zeta(s)$.

Landau (Explicit Formula): For $T \geq T_0$ uniformly in T , we have

$$\psi(x) = x - \sum_{|\gamma| \leq T} \frac{x^{\rho}}{\rho} + O(xT^{-1}(\log xT)^2) + O(\log x)$$

where the sum is over **non-trivial zeros** $\rho = \beta + i\gamma$ of $\zeta(s)$.

$$\text{RH} \implies \text{PNT}.$$

The difference lies in the error term of the approximation to $\pi(x)$.

Without RH (PNT alone):

$$\pi(x) = \text{Li}(x) + o\left(\frac{x}{\log x}\right).$$

With RH:

$$\pi(x) = \text{Li}(x) + O(\sqrt{x} \log x).$$

Similarly,

$$\psi(x) = x + O(\sqrt{x} \log^2 x).$$

$$\begin{aligned}\zeta^2(s) &= \left(\sum_{n_1=1}^{\infty} \frac{1}{n_1^s} \right) \left(\sum_{n_2=1}^{\infty} \frac{1}{n_2^s} \right) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{1}{(n_1 n_2)^s} \\ &= \sum_{n=1}^{\infty} \left(\sum_{\substack{n_1, n_2 \geq 1 \\ n_1 n_2 = n}} 1 \right) \frac{1}{n^s}\end{aligned}$$

and

$$\sum_{\substack{n_1, n_2 \geq 1 \\ n_1 n_2 = n}} 1 = d(n)$$

Generally,

$$\zeta^k(s) = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} \quad \text{for } \Re(s) > 1$$

where

$$d_k(n) := \#\{(n_1, n_2, \dots, n_k) \in \mathbb{N} : n = n_1 n_2 \dots n_k\} \quad \text{for } k \geq 2.$$

Unsolved

$$\sum_{n \leq x} \Lambda(n^2 + 1) = ?$$

Gauss Circle Problem

$$\sum_{\substack{n_1, n_2 \in \mathbb{Z} \\ n_1^2 + n_2^2 \leq x}} 1 = \pi x + E(x); \quad E(x) = O(x^{0.6271})$$

Theorem (with S. K. Singh, 2025¹³)

Let $X \geq 1$, and $\alpha \in \mathbb{Z} \setminus \{0\}$. Then for any $\varepsilon > 0$, we have

$$\sum_{n_1 \sim X} \sum_{n_2 \sim X} d(n_1) d(n_1^2 + \alpha n_2^2) = X^2 R_2(\log X) + O_\varepsilon(X^{2-\frac{1}{2}+\varepsilon}),$$

where $R_2(y)$ is a polynomial of degree 2 in y .

Theorem (with S. K. Singh, 2025¹³)

Let $X \geq 1$, and $\alpha \in \mathbb{Z} \setminus \{0\}$. Then for any $\varepsilon > 0$, we have

$$\sum_{n_1 \sim X} \sum_{n_2 \sim X} d_3(n_1) d(n_1^2 + \alpha n_2^2) = X^2 R_3(\log X) + O_\varepsilon(X^{2-\frac{3}{8}+\varepsilon}),$$

where $R_3(y)$ is a polynomial of degree 3 in y .

¹³H. Chanana and S. K. Singh, *Sum of the $GL(2)$ Fourier coefficients over quadratics with arithmetic weight*, Submitted (2025).

Thanks for your attention!