Hiding in the details — fractals and me
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PiWORKS Seminar
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Outline of talk

01 A bit about me

02 What is ‘dimension’?

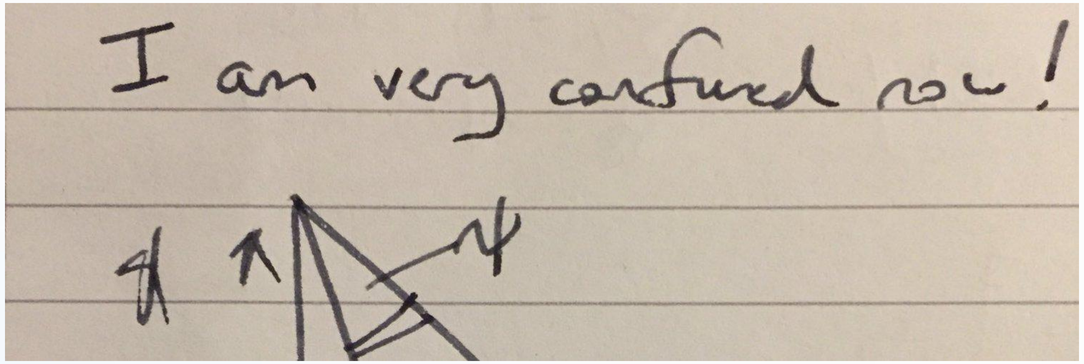
03 Visible sets

04 Monotone sets

A bit (too much) about me

Tacey Coraline O’Neil (she/they)

Born: June 1970; started transition in January 2021.



Mathematical career

Undergrad: University of Bristol, 1988—1991

Postgrad: University College London, 1991—1994

Postdoc: University of St Andrews, 1994—1997

Temporary lectureship: University of Edinburgh, 1997—1999

Joined the Open University in 1999; promoted to Senior Lecturer in 2007.

I work in mathematical analysis, with a particular emphasis on geometric measure theory and fractals.

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Personal

Diagnosed with ADHD (June 2025) and likely autistic (RAADS-R 157, CAT-Q 127)

Life events

Shy teenager: always felt like an alien mimicking the people around me

Figured myself out during PhD but I was scared and thought coming out would be career ending.

Hid and suppressed myself for decades

Increasingly depressed as time went by – very bad in 2019 and 2020

Finally decided only had two options and one was not palatable

Started hormones and coming out to people in January 2021

Divorced September 2022

Affirmation surgery February 2025, GRC May 2025

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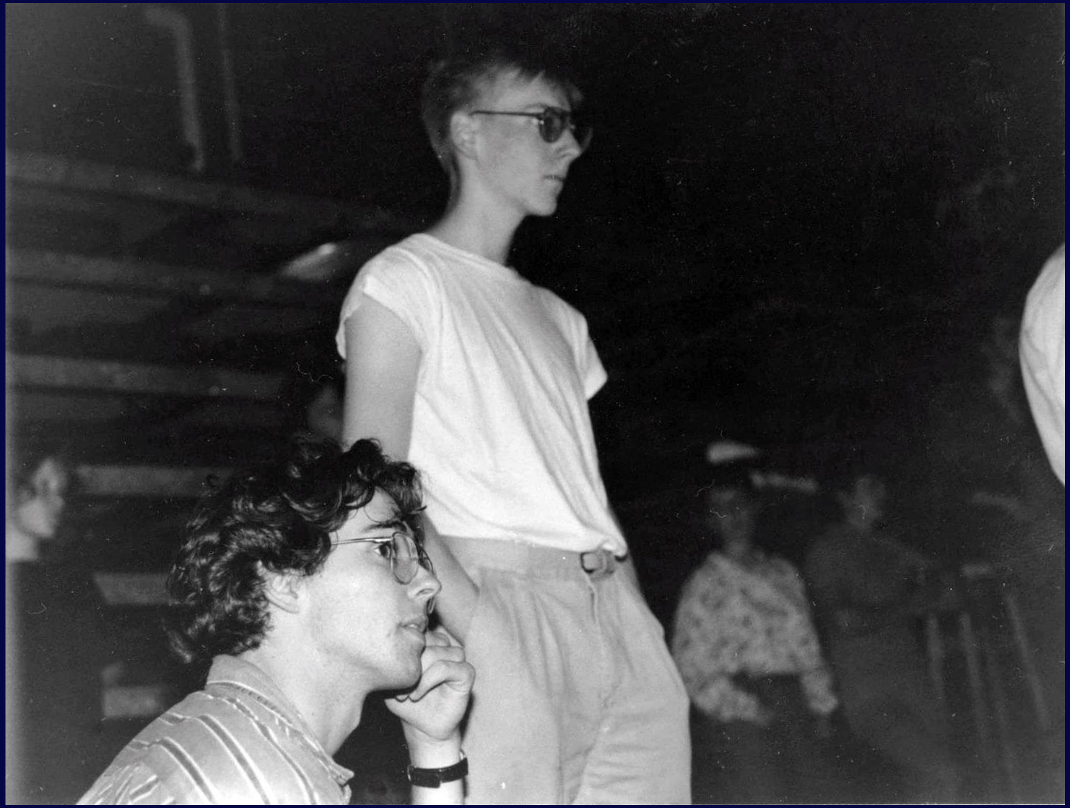
What is ‘dimension’?

For ease, assume that working in plane, $\mathbb{R}^2$, with usual notion of distance in following



What properties should a definition of dimension have?

1. Assigns a real number to any (most) subsets of the space
2. The empty set should have dimension $-\infty$ or 0
3. A singleton set should have dimension 0
4. Gives the right (integer) value to standard shapes
5. *Monotonicity*: if $A \subseteq B$, then dimension of A should at most that of B
6. Gives some indication of how much a set 'fills' the ambient space
7. *Countably stable*: the dimension of a countable collection of sets is the supremum of their individual dimensions

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Common dimension definitions

Almost as many definitions as there are mathematicians interested in dimension

1. Dimension of a vector space
2. Various topological dimensions
3. **(upper and lower) box dimension**
4. (upper and lower) correlation dimension
5. **Hausdorff dimension**
6. Packing dimension
7. Assouad dimension

Box dimension

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Suppose F is a bounded set.

For $\delta > 0$, let $N_\delta(F)$ be the least number of sets of diameter at most δ that cover F.

lower box dimension

$$\underline{\dim}_B(F) = \liminf_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

upper box dimension

$$\overline{\dim}_B(F) = \limsup_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

box dimension

$$\dim_B(F) = \text{ the common value (if it exists)}$$

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many equivalent formulations – see Chapter 2 of *Fractal Geometry* by Falconer (2014)

Hausdorff dimension

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Hausdorff measure

Let F be a set (in the plane).

For $s \geq 0$, define the s-dimensional Hausdorff measure of F to be

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F),$$

where

$$\mathcal{H}^s_\delta(F) = \inf\left\{\sum_{i=1}^{\infty} \left(\mathrm{diam}(U_i)\right)^s : \{U_i\} \text{ is a (countable) } \delta\text{–cover of } F\right\}.$$

Hausdorff dimension

$$\dim_H(F) = \inf\{s : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}$$

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see Chapter 3 of *Fractal Geometry* by Falconer (2014)

Relationship between box and Hausdorff dimension

Let F be a (non-empty bounded) set.

The following chain of inequalities holds:

$$\dim_H(F) \leq \underline{\dim}_B(F) \leq \overline{\dim}_B(F)$$

see Chapter 3 of *Fractal Geometry* by Falconer (2014)

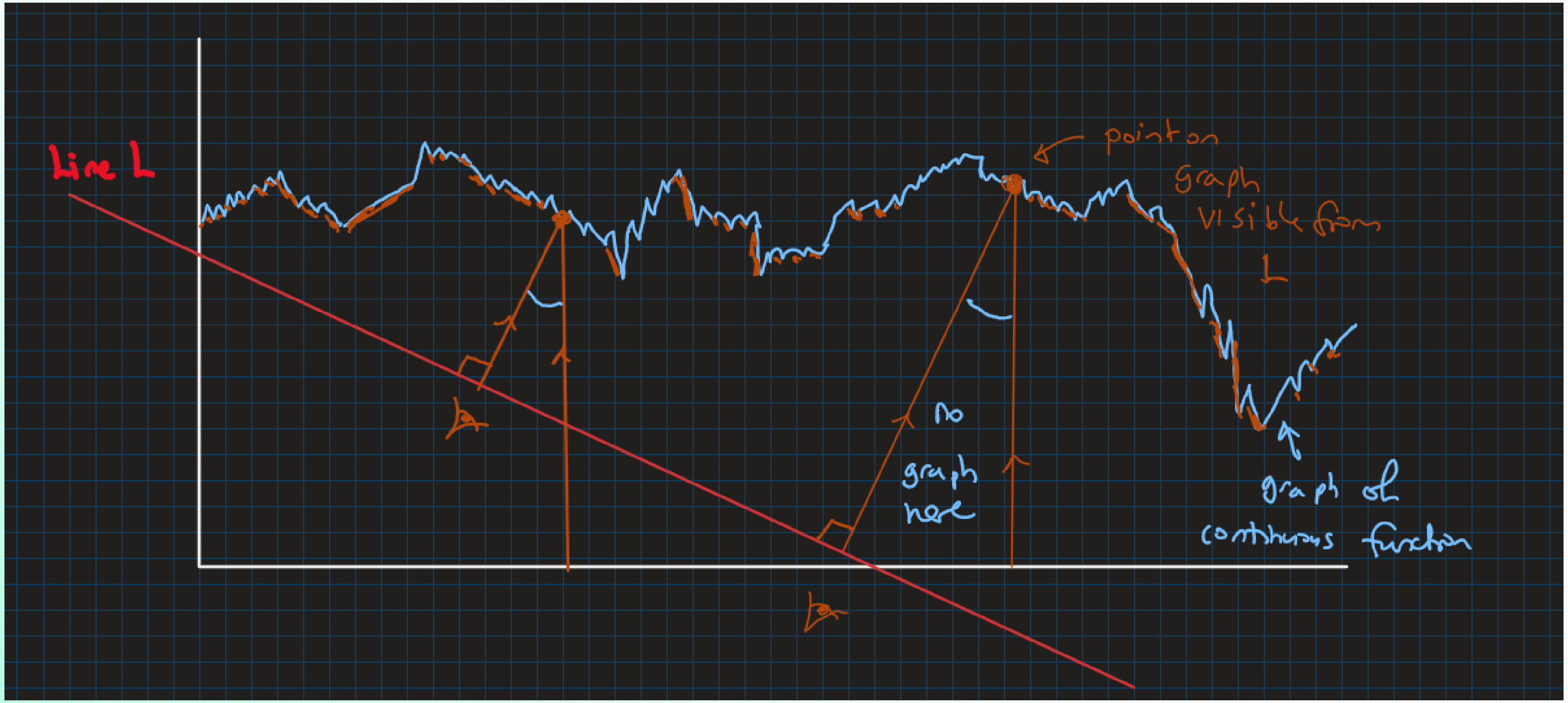
Visible sets

Given a (compact) set in the plane, what is the (expected) dimension of what you can see?

Graphs of continuous functions from $[0,1]$ to $\mathbb{R}$

Well-known(!) that these (compact subsets of the plane) can have Hausdorff dimension 2.

But… if you look at the 'first' point of the graph visible from any line not parallel to the x-axis, then something interesting happens.



Theorem ([O1], 2001)

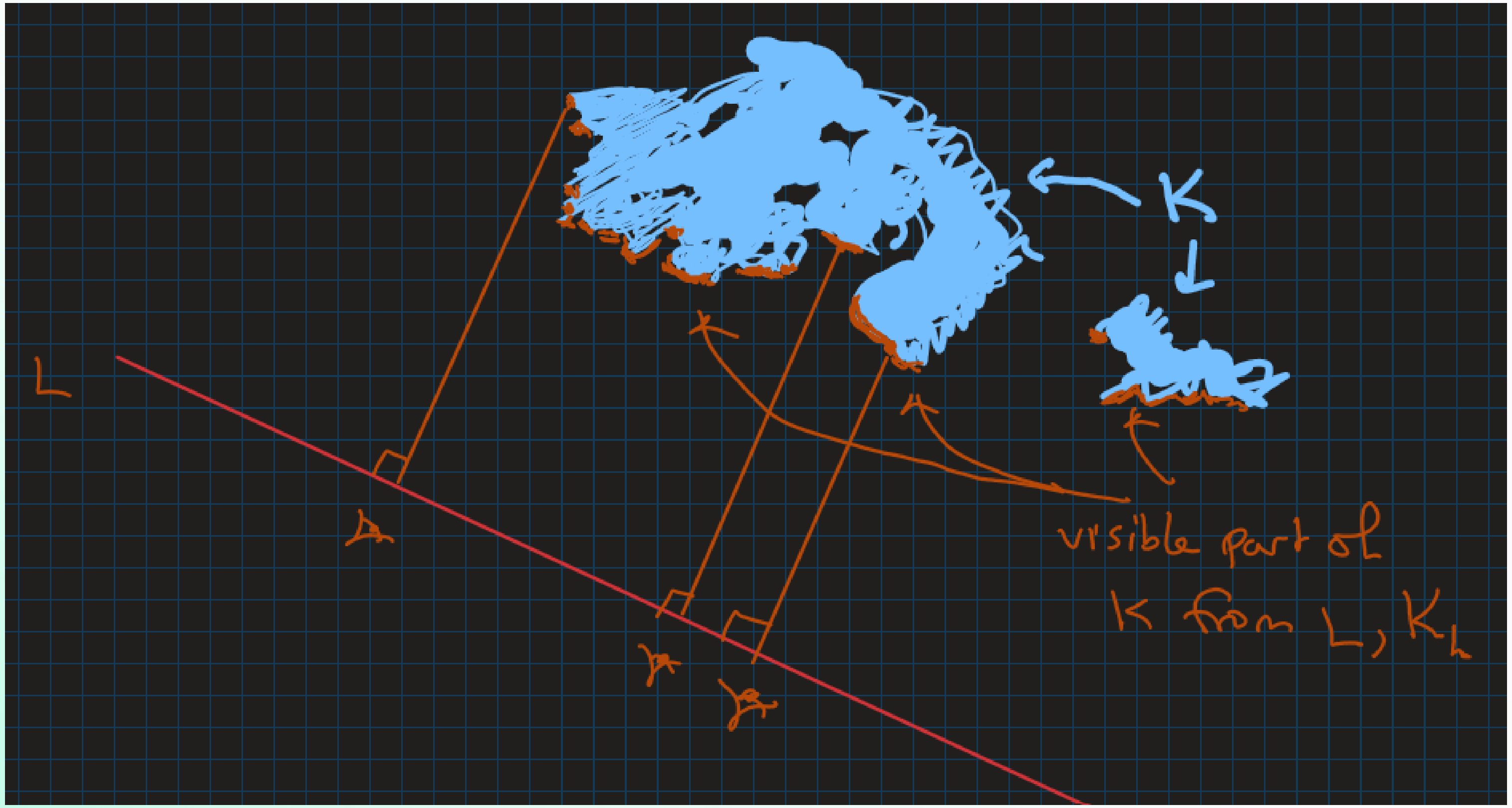
For any graph of a continuous function from $\mathbb{R}$ to $\mathbb{R}$ and any line not parallel to the x-axis, the points of 'first hit' from that line are a rectifiable set and so, in particular, have dimension 1.

An entirely similar result holds if you look at the set of points of 'first hit' of radial rays from a fixed point.

Visible part

First kind: visible part of K from a line L, K_L

Visible from a line



Graphs of continuous functions are special – other sets?

Theorem [JJMO, 2003]

Let K be a quasicircle in the plane of distortion $M \geq 1$.

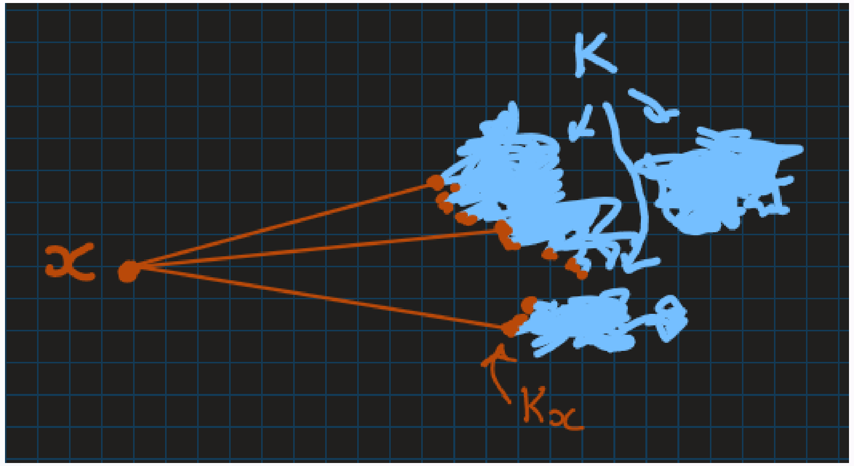
Then the upper box dimension of the visible part of K from all affine lines is at most 1.
That is for all lines L in the plane,

$$\overline{\dim}_B\,(K_L) \leq 1.$$

(A similar result holds for connected self-similar set in the plane.)

Visible part

Second kind: visible part from a point



Visibility from a point

For a compact set in the plane, K, and $x \in \mathbb{R}^2$, we define the visible part of K from x, K_x, by

$$K_x = \{u \in K : [x, u] \cap K = \{u\}\},$$

where $[x, u]$ denotes the line segment that joins x and u.

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Visible parts from a point of continua in the plane

Theorem ([O2], 2007)

If $K \subset \mathbb{R}^2$ is a compact connected set with $\dim_H(K) > 1$, then for (Lebesgue) almost all $x \in \mathbb{R}^2$,

$$\dim_H(K_x) \leq \frac{1}{2} + \sqrt{\dim_H(K) - \frac{3}{4}}.$$

For such a K, we have:

- if $\dim_H(K) = 2$, then for (Lebesgue) almost all x, $\dim_H(K_x) \leq \frac{1}{2}(1 + \sqrt{5})$
- if $\dim_H(K) = 1 + \varepsilon$, then for (Lebesgue) almost all x, $\dim_H(K_x) \leq 1 + \varepsilon - \varepsilon^2 + O(\varepsilon^3)$

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Still an open problem in general

Tuomas Orponen has published more recently on this problem. But still only partial results.

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Monotone sets

How big are sets with a controlled linear ordering?

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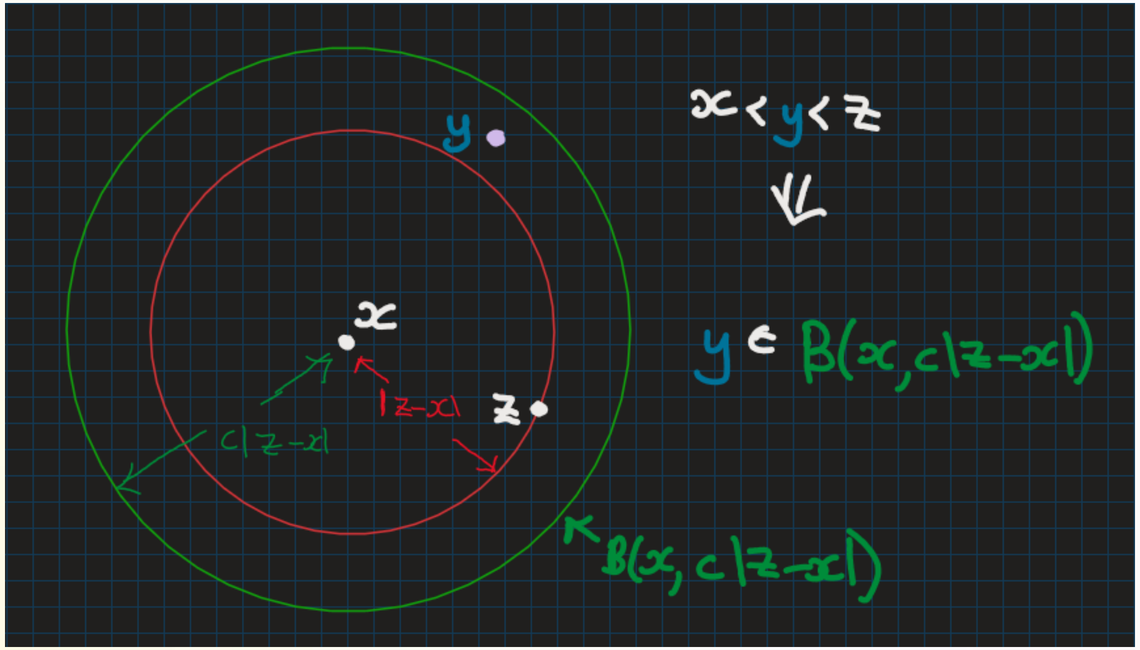
Definition of monotone sets

Definition
Let $c \geq 1$. A (compact) subset K of the plane is c-monotone if there is a linear ordering $<$ on K such that whenever $x, y, z \in K$ with $x < y < z$, then

$$|y - x| \leq c\,|z - x|$$

The definition makes sense in any metric space, see [Z] for an introduction and discussion.



The problem

Suppose that K is a compact c-monotone set for some c near 1.

Is $\dim(K) < 2$?

We don't really care about which dimension is used here: box dimension is fine. We would like some kind of asymptotic estimate.

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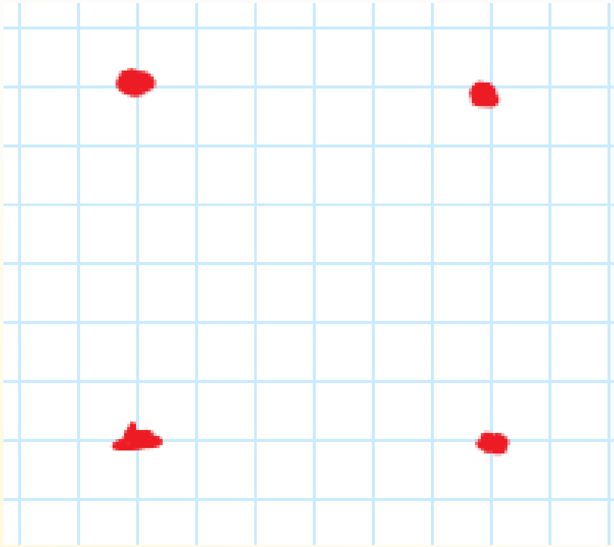
An observation

Lemma

If $c < \sqrt{2}$ and $(K, <)$ is a c-monotone set, then K meets at most three vertices of any square.



label points

1 2 3 4

with

$1 < 2 < 3 < 4$

in ordering

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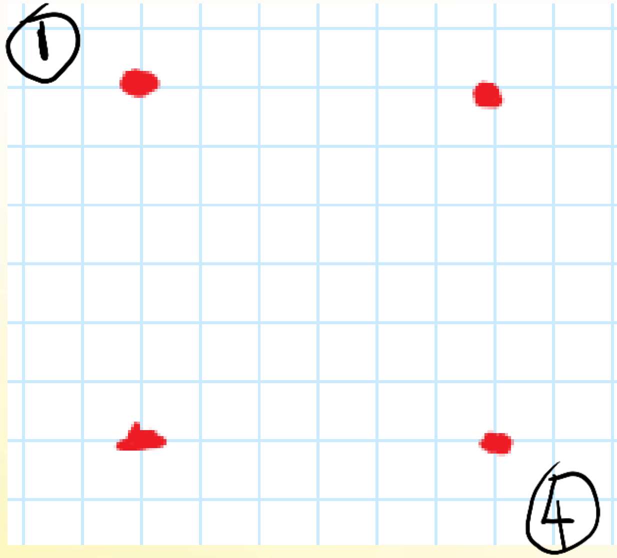
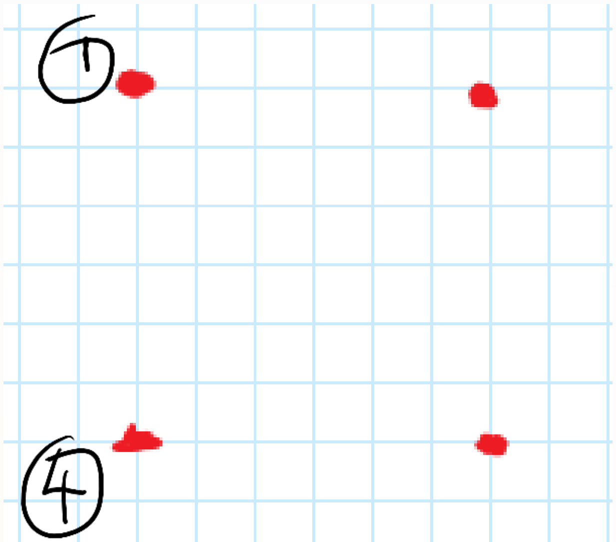


OR



This leads to a box dimension estimate!

Theorem (O'Neil, unpublished)

If $c > 1$, K is a (compact) c-monotone set and c is sufficiently close to 1, then

$$\overline{\dim}_B(K) < 2.$$

The upper bound for the box dimension as $c \to 1$ turns out to be 1.916… so pretty rubbish!

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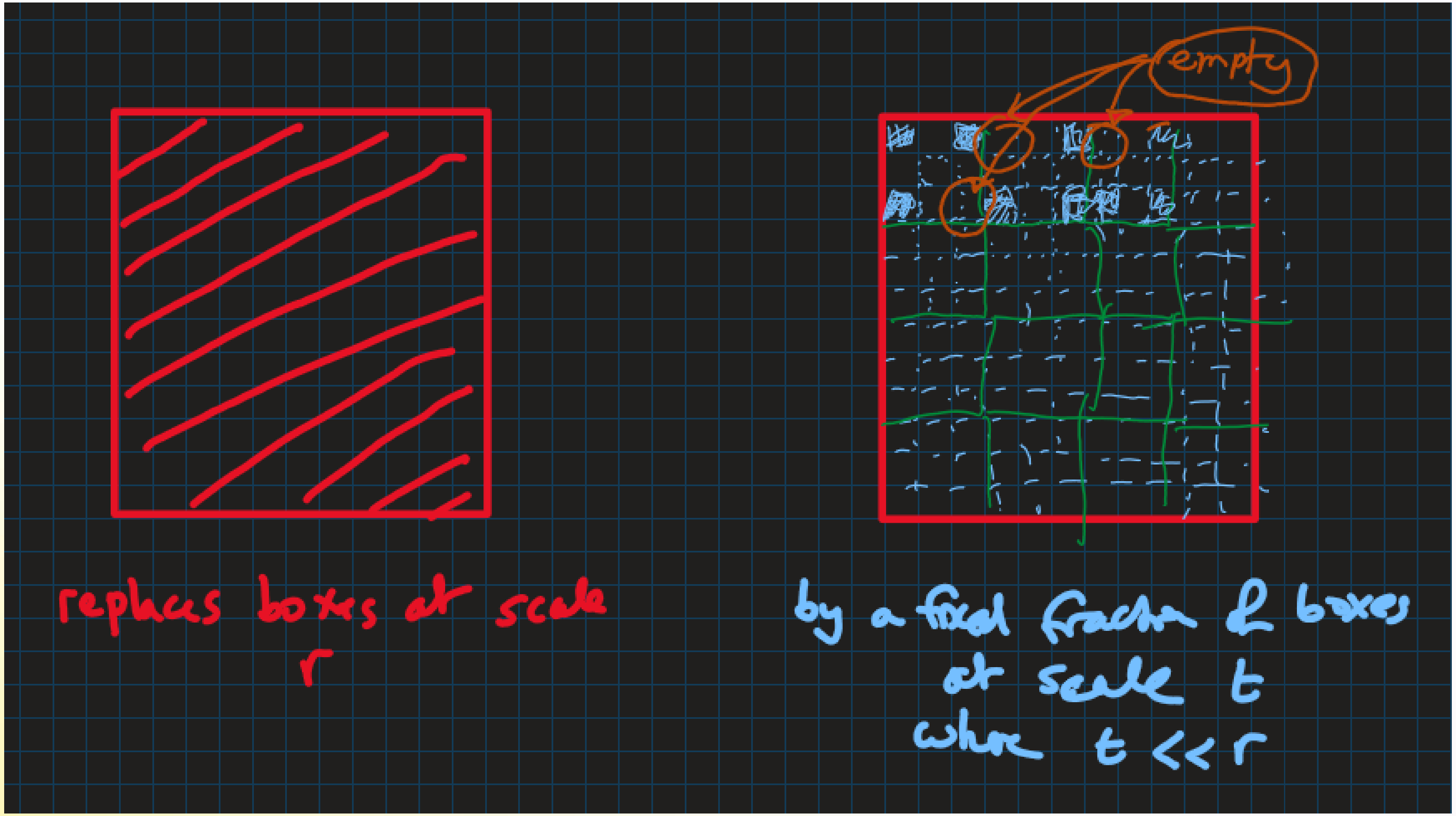
The core idea of the proof

Box dimension is all about counting boxes!



Different shapes?

The 9 vertices of 2×2 square grid meet at most 6 points of a c-monotone set if c is small enough. (And can presumably obtain similar results for other square grids.)

Lemma (O'Neil, unpublished)

Let n be a natural number.
Suppose that K is a c-monotone set for some $c \in \left(1, \sqrt{2/(1+\cos(\pi/n))}\right)$, then K meets each regular $2n$-gon in at most $n+1$ points.

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Find the right approach and get the best result.

We're also lacking in good examples.

What does the von Koch curve, which is c-monotone, tell us?

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References

[JJMO] Järvenpää, Järvenpää, MacManus, O'Neil, Visible parts and dimensions, Nonlinearity

[O1] O'Neil, Graphs of continuous functions from R to R are not purely unrectifiable, Real Analysis Exchange

[O2] O'Neil, The Hausdorff dimension of visible sets of planar continua, Transactions of the American Mathematical Society

[Z] Zindulka, Recent progress in the theory of monotone metric spaces, Real Analysis Exchange

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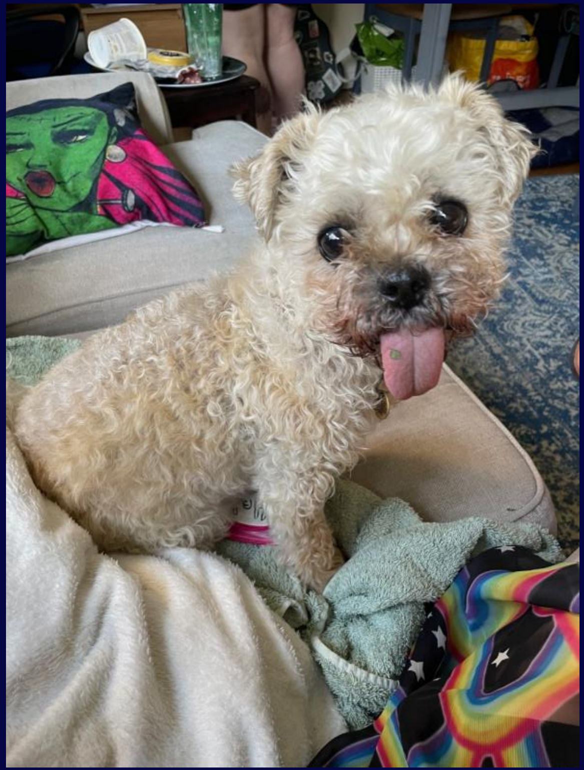

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