Infinite groups: algebra, combinatorics and drawing pictures!

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PiWORKS Seminar
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The University of Manchester

POA

1. Introduction to Geometric Group Theory
2. My complicated relationship with RAAGs
3. Career journey and inspiration...???

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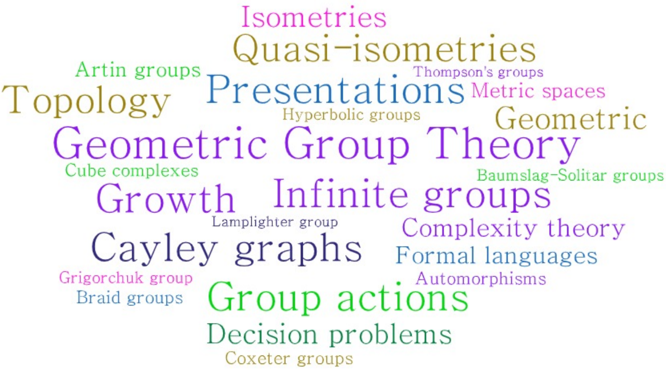
Introduction



Finite groups recap

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4. **Inverses:** for all $x \in G$, there exists a unique $x^{-1} \in G$ such that $x * x^{-1} = x^{-1} * x = e$.

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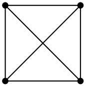
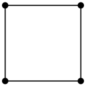
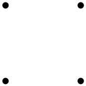


$\mathbb{Z}^4$



$F_2 \times F_2$



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In general, these problems are undecidable.

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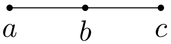
Problem: multiple words over generators can represent the same group element

Words and group elements in RAAGs

$\Gamma:$



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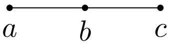

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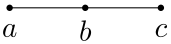

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Question: how do we find *unique* representatives for each group element?

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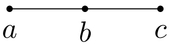

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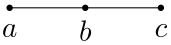

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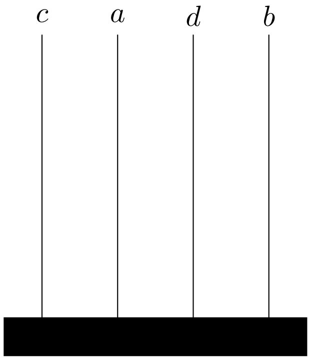
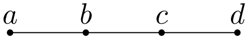
Better answer: by constructing *pilings*.

Pilings





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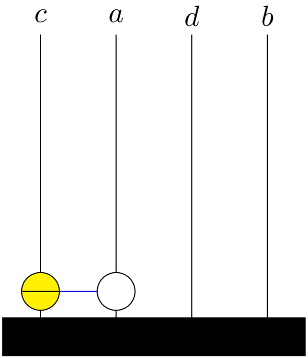

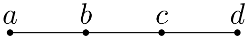

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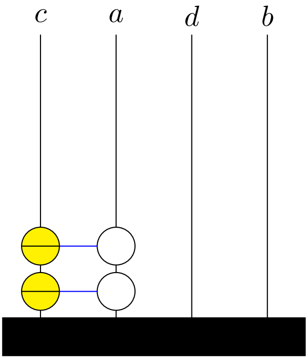

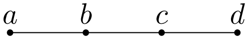

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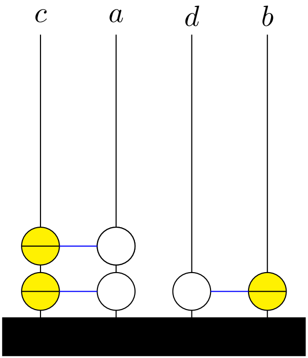

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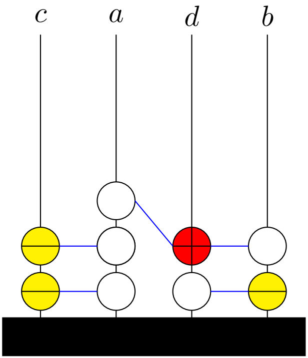

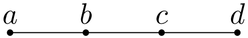

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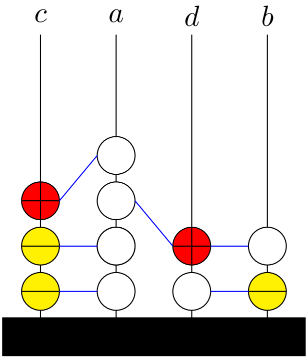

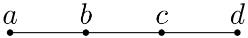

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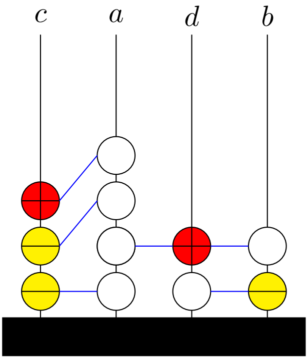

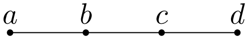

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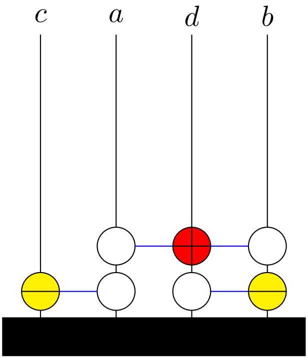

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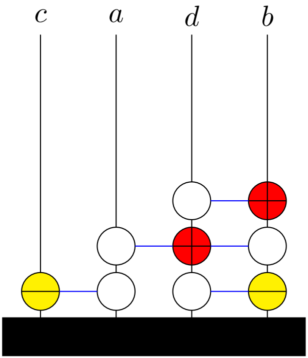

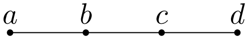

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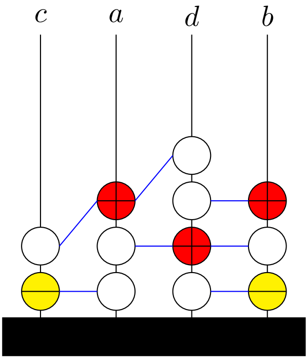

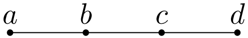

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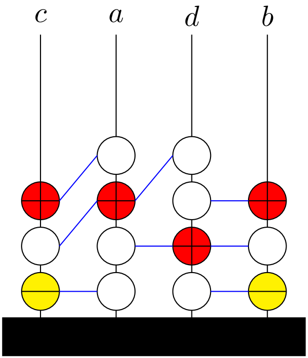

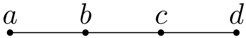

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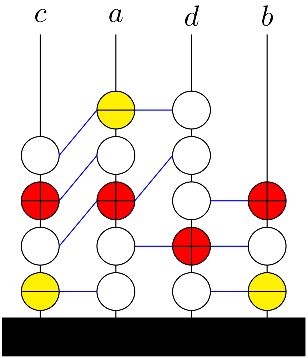

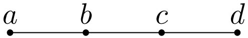

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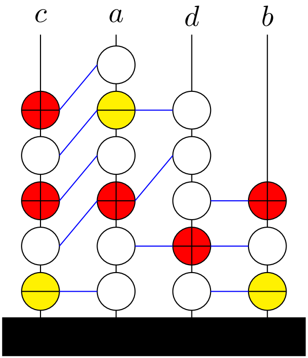

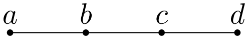

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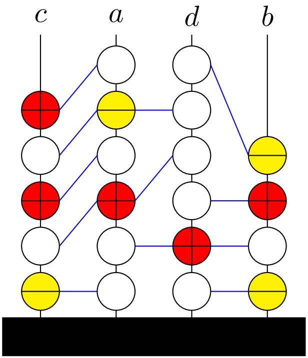

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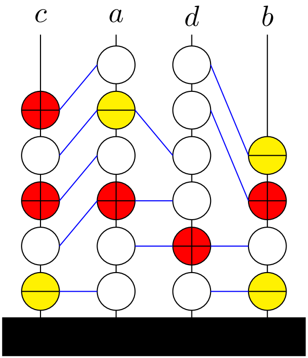

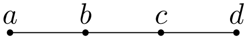

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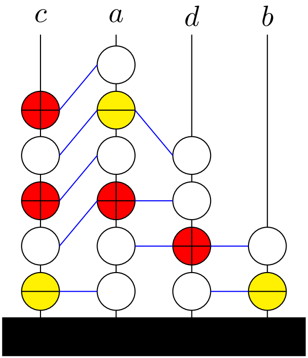

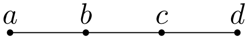

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Let G be a finitely presented group, given by a presentation $\langle X \mid R \rangle$. The **twisted conjugacy problem**(TCP(G)) takes as input two words $u, v \in X^*$ and decides whether there exists $g \in G$ such that

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- Stronger version of $\mathrm{CP}(G)$
- Decidable/undecidable $\mathrm{TCP}(G)$ can lead to decidable/undecidable $\mathrm{CP}(G')$ for *extensions* G' of G, i.e. $G \leq G'$.

Results

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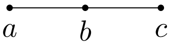
Idea: adapt CGW algorithm for $\mathrm{CP}(A_\Gamma)$

Automorphisms of RAAGs

$\Gamma :$



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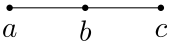

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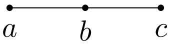

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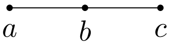

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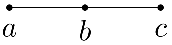

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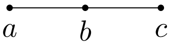

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(Autos of RAAGS: Laurence, Servatius)

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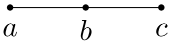

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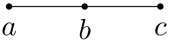

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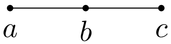

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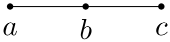

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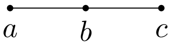

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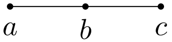

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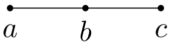

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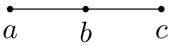

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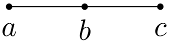

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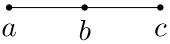

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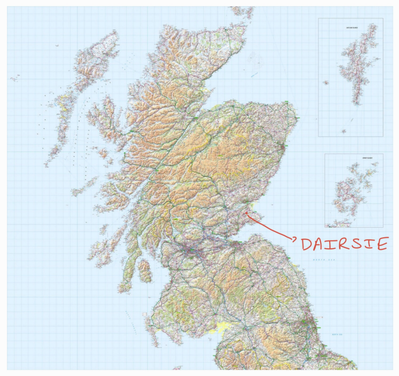
$\Rightarrow$ ϕ-cyclic permutations preserve ϕ-conjugacy.

Other fun things I think about

POA

1. Introduction to Geometric Group Theory
2. My complicated relationship with RAAGs
3. Career journey and inspiration...???

Let's start at the beginning



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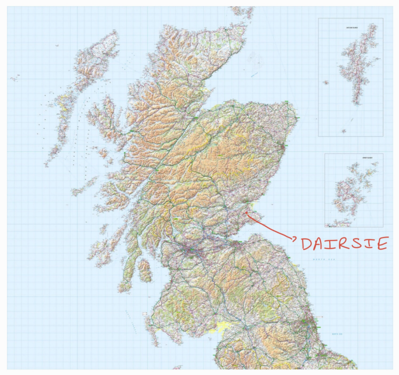

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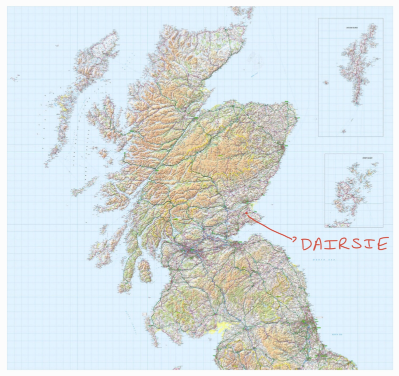

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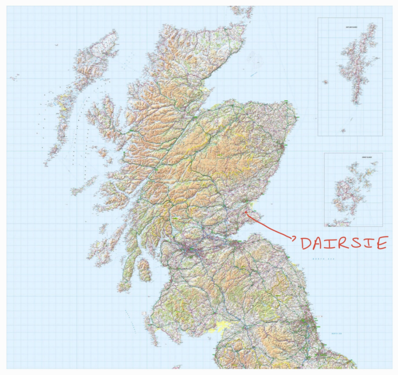

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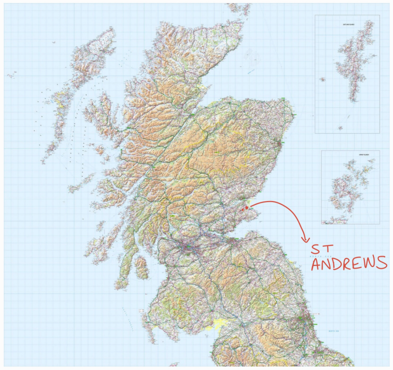
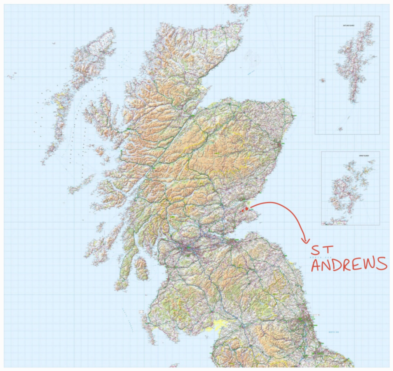
Undergrad



University of
St Andrews

Undergrad



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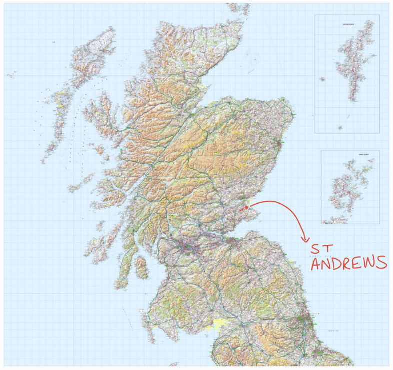

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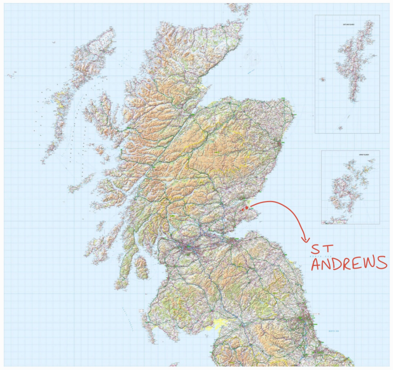

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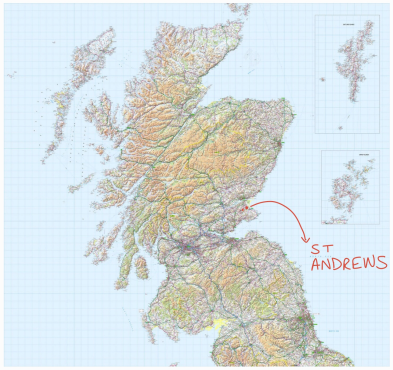

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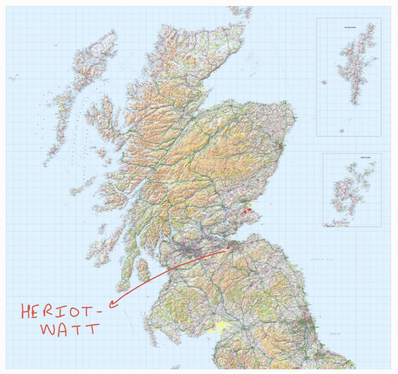

HERIOT WATT UNIVERSITY

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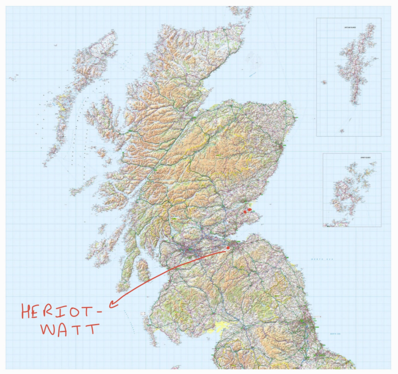

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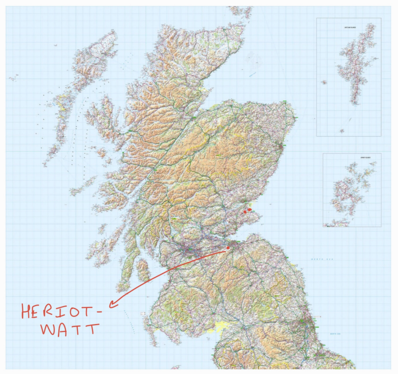

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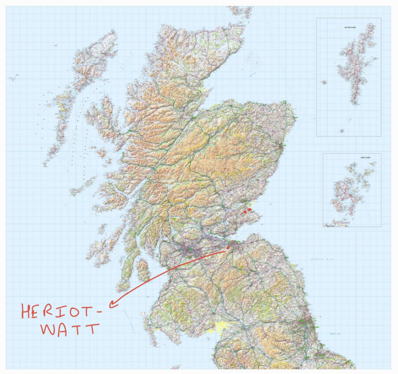

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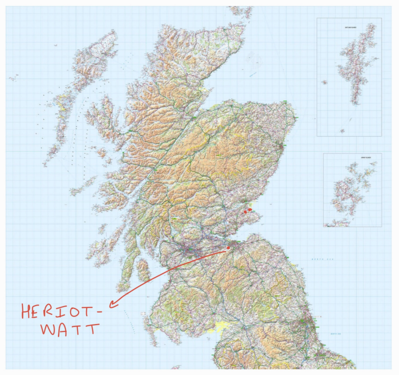

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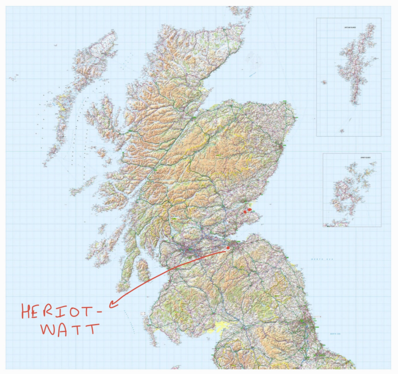

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PhD

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Piscopia initiative

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And now

MANCHESTER 1824
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- **What is right for you?**

References

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Thank you for listening!

Any questions?